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Weighted Pseudo Almost Automorphy of Semilinear Boundary Differential Equations *

Zhinan Xia†

Department of Applied Mathematics,
Zhejiang University of Technology,
Hangzhou, Zhejiang, 310023, China

Abstract

In this paper, we investigate the existence and uniqueness of weighted pseudo almost automorphic mild solutions to semilinear boundary differential equations in Banach space, where the nonlinear perturbation is weighted pseudo almost automorphic type or weighted Stepanov-like pseudo almost automorphic type. As applications, some interesting examples are presented to illustrate the main findings.

Keywords: weighted pseudo almost automorphy; semilinear boundary differential equations; hyperbolic semigroup; sectorial operator.

AMS Subject Classifications: 43A60; 35B15

1 Introduction

The notation of almost automorphy, introduced by Bochner [4] is related to and more general than almost periodicity. Since then, this pioneer work attracts more and more attentions and is substantially extended in several different directions. For more details about this topic, we refer to the recent books [16, 17, 20], where the authors gave an important overview about the theory of almost automorphic functions and their applications to differential equations. Recently, a new more general type of almost automorphy called weighted pseudo almost automorphy is proposed by Blot et al [3], which generalize various extension of almost automorphy and almost periodicity such as asymptotic almost automorphy (periodicity) [11, 15], pseudo almost automorphy (periodicity) [23, 24], weighted pseudo almost periodicity [7], and so on. However, literatures concerning weighted pseudo almost automorphy, especially in the Stepanav case are very few [14, 26, 27].

Because of the significance and applications, varieties of problems of boundary differential equations have been addressed by several researchers. They are widely and efficiently used to

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†Corresponding author. E-mail: xiazn299@zjut.edu.cn

describe many phenomena that arise in physical, biology and other subjects. Some properties of the solutions have been studied in several contexts. Recently, the almost periodicity and almost automorphy of boundary differential equations have been extensively explored in the literatures [1, 2, 22]. However, to the best of our knowledge, the weighted pseudo almost automorphic solutions to semilinear boundary differential equations with weighted pseudo almost automorphic (or weighted Stepanov-like pseudo almost automorphic) coefficients have not been treated in the literatures yet. This is one of the key motivations of this study.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented. Section 3 is divided into two parts. In the first one, Section 3.1, we investigate the existence and uniqueness of weighted pseudo almost automorphic solutions to semilinear boundary differential equations with weighted pseudo almost automorphic coefficients. In the second part, Section 3.2, for the Stepanov-like pseudo almost automorphic perturbation, we study the weighted pseudo almost automrophy of semilinear boundary differential equations. In Section 4, an application to partial differential equation is given.

2 Preliminaries and Basic Results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two Banach spaces and $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integers, real and complex numbers, respectively. In order to facilitate the discussion below, we further introduce the following notations:

- $BC(\mathbb{R}, X)$ (resp. $BC(\mathbb{R}\times Y, X)$: the Banach space of bounded continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R} \times Y$ to X) with the supremum norm.
- $C(\mathbb{R}, X)$ (resp. $C(\mathbb{R} \times Y, X)$): the set of continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R} \times Y$ to X).
- $B(X, Y)$: the Banach space of bounded linear operators from X to Y endowed with the operator topology.
- $L^p(\mathbb{R}, X)$: the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable functions $f : \mathbb{R} \to X$ such that $\|f\| \in L^p(\mathbb{R}, \mathbb{R})$.
- $L^p_{loc}(\mathbb{R}, X)$: stand for the space of all classes of equivalence of measurable functions $f : \mathbb{R} \to X$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, X)$.

2.1 Extrapolation Banach Space

Definition 2.1. [10] A linear operator $A : D(A) \subset X \to X$ is said to be sectorial if the following hold: there exist constants $\omega \in \mathbb{R}, \theta \in (\pi/2, \pi)$ and $\widetilde{M} > 0$ such that

$$\begin{aligned} &\rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}, \\ &\|R(\lambda, A)\| \le \frac{\widetilde{M}}{|\lambda - \omega|}, \quad \lambda \in S_{\theta,\omega}, \end{aligned} \tag{2.1}$$

where $R(\lambda, A) := (\lambda I - A)^{-1}$ for each $\lambda \in S_{\theta,\omega}$.

For $\alpha \in (0,1)$, we make use of the real interpolation space

$$X_\alpha := \overline{D(A)}^{\|\cdot\|_\alpha},$$

which is a Banach space endowed with the norm

$$\|x\|_\alpha := \sup_{\lambda>0} \|\lambda^\alpha (A-\omega)R(\lambda, A-\omega)x\|.$$

For convenience, we further write $X_0 := X$, $X_1 := D(A)$ and $\|x\|_0 = \|x\|$, $\|x\|_1 = \|(A-\omega)x\|$. On $\widehat{X} := \overline{D(A)}$, we introduce a new norm:

$$\|x\|_{-1} = \|(\omega - A)^{-1}x\|, \quad x \in X.$$

The completion of $(\widehat{X}, \|x\|_{-1})$ is called the extrapolation space of X associated with A and will be denoted by X_{-1}, then A has a unique continuous extension $A_{-1} : \widehat{X} \to X_{-1}$. Since $T(t)$ commutes with the operator resolvent $R(\omega, A)$, the extension of $T(t)$ to X_{-1} exists and defines an analytic semigroup $(T_{-1}(t))_{t\geq 0}$ which is generated by A_{-1} with $D(A_{-1}) = \widehat{X}$. As above, we define the space

$$X_{\alpha-1} := (X_{-1})_\alpha = \overline{\widehat{X}}^{\|\cdot\|_{\alpha-1}}$$

with

$$\|x\|_{\alpha-1} = \sup_{\lambda>0} \|\lambda^\alpha R(\lambda, A_{-1}-\omega)x\|.$$

The restriction $A_{\alpha-1} : X_\alpha \to X_{\alpha-1}$ of A_{-1} generates the analytic semigroup $(T_{\alpha-1}(t))_{t\geq 0}$ on $X_{\alpha-1}$ which is the extension of $T(t)$ to $X_{\alpha-1}$. Observe that $\omega - A_{\alpha-1} : X_\alpha \to X_{\alpha-1}$ is an isometric isomorphism. We will frequently use the continuous embeding

$$D(A) \hookrightarrow X_\beta \hookrightarrow D((\omega-A)^\alpha) \hookrightarrow X_\alpha \hookrightarrow X,$$

$$X \hookrightarrow X_{\beta-1} \hookrightarrow D((\omega-A_{-1})^\alpha) \hookrightarrow X_{\alpha-1} \hookrightarrow X_{-1},$$

for all $0 < \alpha < \beta < 1$.

Definition 2.2. [10] An analytic semigroup $(T(t))_{t\geq 0}$ is said to be hyperbolic if it satisfies the following properties:

(i) there exist two subspace X_s (the stable space) and X_u (the unstable space) of X such that $X = X_s \oplus X_u$;

(ii) $T(t)$ is defined on X_u, $T(t)X_u \subset X_u$, and $T(t)X_s \subset X_s$ for all $t \geq 0$.

(iii) there exist constants $M, \delta > 0$ such that

$$\|T(t)P_s\| \leq Me^{-\delta t}, \quad t \geq 0, \qquad \|T(t)P_u\| \leq Me^{\delta t}, \quad t \leq 0,$$

where P_s and P_u are the projection onto X_s and X_u respectively.

Recall that an analytic semigroup $(T(t))_{t\geq 0}$ is hyperbolic if and only if $\sigma(A) \cap i\mathbb{R} = \emptyset$.

Lemma 2.1. *[2] For $x \in X_{\alpha-1}$ and $0 \leq \beta \leq 1$, $0 < \alpha < 1$, then the following assertions hold:*

(i) *there is a constant c such that*

$$\|T_{\alpha-1}(t)P_{u,\alpha-1}x\|_\beta \leq ce^{\delta t}\|x\|_{\alpha-1} \quad for \quad t \leq 0; \tag{2.2}$$

(ii) *there is a constant m such that for $0 < \alpha - \widetilde{\varepsilon} < 1$*

$$\|T_{\alpha-1}(t)P_{s,\alpha-1}x\|_\beta \leq me^{-\gamma t}t^{\alpha-\beta-\widetilde{\varepsilon}-1}\|x\|_{\alpha-1} \quad for \quad t \geq 0. \tag{2.3}$$

2.2 Weighted Pseudo Almost Automorphy

First, let us recall some definitions of almost automorphic function and weight pseuso almost automorphic function.

Definition 2.3. (Bochner [4]) A function $f \in C(\mathbb{R}, X)$ is said to be almost automorphic in Bochner's sense if for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ is well defined for each $t \in \mathbb{R}$, and $\lim_{n\to\infty} g(t-s_n) = f(t)$ for each $t \in \mathbb{R}$.

Almost automorphic functions (denoted by $AA(\mathbb{R}, X)$) constitute a Banach space when it is endowed with the sup norm. They naturally generalize the concept of (Bochner) almost periodic functions.

Lemma 2.2. *[16] If $f, f_1, f_2 \in AA(\mathbb{R}, X)$, then*
(i) $f_1 + f_2 \in AA(\mathbb{R}, X)$,
(ii) $\lambda f \in AA(\mathbb{R}, X)$ for any scalar λ,
(iii) $f_\alpha \in AA(\mathbb{R}, X)$ where $f_\alpha : \mathbb{R} \to X$ is defined by $f_\alpha(\cdot) := f(\cdot + \alpha)$,
(iv) the range $\Re_f := \{f(t) : t \in \mathbb{R}\}$ is relatively compact in $\mathbb{X}$, thus f is bounded in norm.
(v) if $f_n \to f$ uniformly on $\mathbb{R}$ where each $f_n \in AA(\mathbb{R}, X)$, then $f \in AA(\mathbb{R}, X)$ too.

Let U be the set of all functions $\rho : \mathbb{R} \to (0, \infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $T > 0$ and each $\rho \in U$, set

$$\mu(T, \rho) := \int_{-T}^{T} \rho(t)\mathrm{d}t.$$

Define

$$U_\infty := \{\rho \in U : \lim_{T\to\infty} \mu(T, \rho) = \infty\}, \quad U_B := \{\rho \in U_\infty : \rho \text{ is bounded and} \inf_{x\in\mathbb{R}} \rho(x) > 0\}.$$

It is clear that $U_B \subset U_\infty \subset U$.

For $\rho \in U_\infty$, define

$$PAA_0(\mathbb{R}, X, \rho) := \left\{ f \in BC(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \rho(t)\|f(t)\|dt = 0 \right\}.$$

$$PAA_0(\mathbb{R} \times X, X, \rho) := \Bigg\{ f \in BC(\mathbb{R} \times X, X) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \rho(t)\|f(t, u)\|dt = 0$$
$$\text{uniformly in } u \in X \Bigg\}.$$

Definition 2.4. [3] Let $\rho \in U_\infty$. A function $f \in C(\mathbb{R}, X)$ (resp. $C(\mathbb{R} \times X, X)$) is called weighted pseudo almost automorphic if it can be decomposed as $f = g + \varphi$, where $g \in AA(\mathbb{R}, X)$ (resp. $AA(\mathbb{R} \times X, X)$) and $\varphi \in PAA_0(\mathbb{R}, X, \rho)$ (resp. $PAA_0(\mathbb{R} \times X, X, \rho)$). Denote by $WPAA(\mathbb{R}, X, \rho)$ (resp. $WPAA(\mathbb{R} \times X, X, \rho)$) the set of such functions.

Definition 2.5. Let $\rho_1, \rho_2 \in U_\infty$. ρ_1 is said to be equivalent to ρ_2 (i.e., $\rho_1 \sim \rho_2$) if $\frac{\rho_1}{\rho_2} \in U_B$.

It is trivial to show that " $\sim$ " is a binary equivalence relation on U_∞. The equivalence class of a given weight $\rho \in U_\infty$ which is denoted by $cl(\rho) = \{\varrho \in U_\infty : \rho \sim \varrho\}$. It is clear that $U_\infty = \bigcup_{\rho \in U_\infty} cl(\rho)$.

Let $\rho \in U_\infty, s \in \mathbb{R}$, defined ρ_s by $\rho_s(t) = \rho(t+s)$ for $t \in \mathbb{R}$ and

$$U_T = \{\rho \in U_\infty : \rho \sim \rho_s \ \text{ for each } \ s \in \mathbb{R}\}.$$

It is trivial to see that U_T contains various kinds of weights such as $1, (1+t^2)/(2+t^2), e^t$, and $1+|t|^n$ with $n \in \mathbb{N}$ et al.

It is obvious that $WPAA(\mathbb{R}, X, \rho)$ (resp. $WPAA(\mathbb{R} \times X, X, \rho)$), $\rho \in U_T$ is a Banach space when endowed with the supremum norm $\|\cdot\|$.

Lemma 2.3. *[25] $PAA_0(\mathbb{R}, X, \rho)$ with $\rho \in U_T$ is translation invariant, that is, $\varphi \in PAA_0(\mathbb{R}, X, \rho)$ and $s \in \mathbb{R}$ implies that $\varphi(\cdot - s) \in PAA_0(\mathbb{R}, X, \rho)$.*

2.3 Weighted Stepanov-like Pseudo Almost Automorphy

Let $p \in [1, \infty)$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to X$ such that $f^b \in L^\infty(\mathbb{R}, L^p([0,1]; X))$, where f^b is the Bochner transfom of f defined by $f^b(t,s) := f(t+s), t \in \mathbb{R}, s \in [0,1]$. $BS^p(\mathbb{R}, X)$ is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \mathrm{d}\tau \right)^{\frac{1}{p}}.$$

It is obvious that $L^p(\mathbb{R}, X) \subset BS^p(\mathbb{R}, X) \subset L^p_{loc}(\mathbb{R}, X)$ and $BS^p(\mathbb{R}, X) \subset BS^q(\mathbb{R}, X)$ for $p \geq q \geq 1$.

Definition 2.6. [8] The space $S^pAA(\mathbb{R}, X)$ of Stepanov-like almost automorphic functions (or S^p-almost automorphic functions) consists of all $f \in BS^p(\mathbb{R}, X)$ such that $f^b \in AA(\mathbb{R}, L^p([0,1], X))$.

In other words, a function $f \in L^p_{loc}(\mathbb{R}, X)$ is said to be Stepanov-like almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p([0,1], X)$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$, there exist a subsequence $(s_n)_{n \in \mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R}, X)$ such that

$$\lim_{n \to \infty} \left(\int_t^{t+1} \|f(s+s_n) - g(s)\|^p ds \right)^{\frac{1}{p}} = 0, \quad \lim_{n \to \infty} \left(\int_t^{t+1} \|g(s-s_n) - f(s)\|^p ds \right)^{\frac{1}{p}} = 0$$

pointwisely on $\mathbb{R}$. The collection of all such functions will be denoted by $S^pAA(\mathbb{R}, X)$.

It is clear that if $1 \leq p < q < \infty$, $f \in L^q_{loc}(\mathbb{R}, X)$ is S^q-almost automorphic, then f is S^p-almost automorphic. Also if $f \in AA(\mathbb{R}, X)$, then f is S^p-almost automorphic for any $1 \leq p < \infty$.

Definition 2.7. [8] A function $f : \mathbb{R} \times X \to X, (t,u) \to f(t,u)$ with $f(\cdot, u) \in L^p_{loc}(\mathbb{R}, X)$ for each $u \in X$ is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly for $u \in X$ if for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g:\mathbb{R}\times X\to X$ with $g(\cdot,u)\in L^p_{loc}(\mathbb{R},X)$ such that

$$\lim_{n\to\infty}\left(\int_0^1\|f(t+s+s_n,u)-g(t+s,u)\|^p ds\right)^{\frac{1}{p}}=0,$$

and

$$\lim_{n\to\infty}\left(\int_0^1\|g(t+s-s_n,u)-f(t+s,u)\|^p ds\right)^{\frac{1}{p}}=0,$$

for each $t\in\mathbb{R}$ and for each $u\in X$. We denote by $S^pAA(\mathbb{R}\times X,X)$ the set of all such functions.

Definition 2.8. Let $\rho\in U_\infty$. A function $f\in BS^p(\mathbb{R},X)$ is said to be weighted Stepanov-like pseudo almost automorphic (or weighted S^p-pseudo almost automorphic) if it can be decomposed as $f=g+\varphi$, where $g^b\in AA(\mathbb{R},L^p([0,1],X))$ and $\varphi^b\in PAA_0(\mathbb{R},L^p([0,1],X),\rho)$. Denote by $S^pWPAA(\mathbb{R},X,\rho)$ the collection of such functions.

In other words, a function $f\in L^p_{loc}(\mathbb{R},X)$ is said to be weighted S^p-pseudo almost automorphic if its Bochner transform $f^b:\mathbb{R}\to L^p([0,1],X)$ is weighted pseudo almost automorphic in the sense that there exist two functions $g,\varphi:\mathbb{R}\to X$ such that $f=g+\varphi$, where $g^b\in AA(\mathbb{R},L^p([0,1],X))$ and $\varphi^b\in PAA_0(\mathbb{R},L^p([0,1],X),\rho)$, i.e.,

$$\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\left(\int_t^{t+1}\|\varphi(\sigma)\|^p d\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0.$$

Definition 2.9. Let $\rho\in U_\infty$. A function $f:\mathbb{R}\times X\to X$, $(t,u)\to f(t,u)$ with $f(\cdot,u)\in BS^p(\mathbb{R},X)$ for each $u\in X$ is said to be weighted S^p-pseudo almost automorphic if it can be decomposed as $f=g+\varphi$, where $g^b\in AA(\mathbb{R}\times X,L^p([0,1],X))$ and $\varphi^b\in PAA_0(\mathbb{R}\times X,L^p([0,1],X),\rho)$. The collection of such functions will be denoted by $S^pWPAA(\mathbb{R}\times X,X,\rho)$.

Theorem 2.1. [21] *Assume that* $\rho\in U_\infty, f=g+\varphi\in S^pWPAA(\mathbb{R}\times X,X,\rho)$ *with* $g^b\in AA(\mathbb{R}\times X,L^p([0,1],X)),\varphi^b\in PAA_0(\mathbb{R}\times X,L^p([0,1],X),\rho)$ *and*

(i) there exist constants $L_f,L_g>0$ *such that*

$$\|f(t,u)-f(t,v)\|\le L_f\|u-v\|,\quad \|g(t,u)-g(t,v)\|\le L_g\|u-v\|,\quad u,v\in X,\quad t\in\mathbb{R}.$$

(ii) $h=\alpha+\beta\in S^pWPAA(\mathbb{R},X,\rho)$ *with* $\alpha^b\in AA(\mathbb{R},L^p([0,1],X))$, $\beta^b\in PAA_0(\mathbb{R},L^p([0,1],X),\rho)$ *and* $K=\overline{\{\alpha(t):t\in\mathbb{R}\}}$ *is compact in* X.

Then $f(\cdot,h(\cdot))\in S^pWPAA(\mathbb{R},X,\rho)$.

Theorem 2.2. [21] *Assume that* $\rho\in U_\infty,p>1,f=g+\varphi\in S^pWPAA(\mathbb{R}\times X,X,\rho)$ *with* $g^b\in AA(\mathbb{R}\times X,L^p([0,1],X)),\varphi^b\in PAA_0(\mathbb{R}\times X,L^p([0,1],X),\rho)$ *and*

(i) there exist nonnegative functions $L_f,L_g\in S^rAA(\mathbb{R},\mathbb{R})$ *with* $r\ge\max\{p,p/(p-1)\}$ *such that*

$$\|f(t,u)-f(t,v)\|\le L_f(t)\|u-v\|,\quad \|g(t,u)-g(t,v)\|\le L_g(t)\|u-v\|,\quad u,v\in X,t\in\mathbb{R}.$$

(ii) $h=\alpha+\beta\in S^pWPAA(\mathbb{R},X,\rho)$ *with* $\alpha^b\in AA(\mathbb{R},L^p([0,1],X))$, $\beta^b\in PAA_0(\mathbb{R},L^p([0,1],X),\rho)$ *and* $K=\overline{\{\alpha(t):t\in\mathbb{R}\}}$ *is compact in* X.

Then there exists a $q\in[1,p)$ *such that* $f(\cdot,h(\cdot))\in S^qWPAA(\mathbb{R},X,\rho)$.

3 Existence and Uniqueness of $WPAA$ Solutions to (3.1)

Consider the semilinear boundary differential equations

$$\begin{cases} x'(t) = A_m x(t) + f(t, x(t)), & t \in \mathbb{R}, \\ Lx(t) = g(t, x(t)), & t \in \mathbb{R}. \end{cases} \tag{3.1}$$

The first equation stands in a Banach space $(X, \|\cdot\|)$ and the second one is in the boundary space ∂X, $(A_m, D(A_m))$ is a densely defined linear operator on X, $L : D(A_m) \to \partial X$ is a bounded linear operator, and f, g are continuous functions.

In this section, we make the following assumptions.

(H_1) There exists a new norm $|\cdot|$ which makes the domain $D(A_m)$ complete and then denoted by X_m. The space X_m is continuous embedded in X and $A_m \in B(X_m, X)$.

(H_2) The restriction $A := A_m|_{ker(L)}$ is sectorial operator such that $\sigma(A) \cap i\mathbb{R} = \emptyset$.

(H_3) $L \in B(X_m, \partial X)$ is surjective.

(H_4) $X_m \hookrightarrow X_\alpha$ for some $0 < \alpha < 1$.

Definition 3.1. A mild solution of (3.1) is a continuous function $x : \mathbb{R} \to X$ satisfying

$$(i) \int_s^t x(\tau)d\tau \in X_m, \quad (ii)\ x(t) - x(s) = A_m \int_s^t x(\tau)d\tau + \int_s^t f(\tau, x(\tau))d\tau,$$

$$(iii)\ L \int_s^t x(\tau)d\tau = \int_s^t g(\tau, x(\tau))d\tau,$$

for all $t \geq s, t, s \in \mathbb{R}$.

As in [2], we transform (3.1) to the equivalent semilinear differential equations

$$x'(t) = A_{\alpha-1}x(t) + f(t, x(t)) - A_{\alpha-1}L_0 g(t, x(t)), \quad t \in \mathbb{R}, \tag{3.2}$$

where $L_0 := (L|_{ker(A_m)})^{-1}$.

3.1 Weighted Pseudo Almost Automorphic Perturbation

In this subsection, we deal with the case that the nonlinear perturbation in (3.1) is weighted pseudo almost automorphic, i.e. the following condition is satisfied:

(H_5) $f \in WPAA(\mathbb{R} \times X_\beta, X, \rho), g \in WPAA(\mathbb{R} \times X_\beta, \partial X, \rho), \rho \in U_T$ for $0 \leq \beta < \alpha$.

We study the existence and uniqueness of weighted pseudo almost automorphic solutions for the inhomogeneous differential equations

$$x'(t) = A_{\alpha-1}x(t) + h(t, x(t)), \quad t \in \mathbb{R}. \tag{3.3}$$

where the function $h : \mathbb{R} \times X_\beta \to X_{\alpha-1}$ satisfies the globally Lipschitzian condition, i.e., there exists a constant $k > 0$ such that

$$\|h(t, x) - h(t, y)\|_{\alpha-1} \leq k\|x - y\|_\beta \quad \text{for all } x, y \in X_\beta, t \in \mathbb{R}.$$

Definition 3.2. A mild solution of (3.3) is a continuous function $x : \mathbb{R} \to X_\beta$ satisfying

$$x(t) = T(t-s)x(s) + \int_s^t T_{\alpha-1}(t-\tau)h(\tau, x(\tau))d\tau, \tag{3.4}$$

for all $t \geq s, t, s \in \mathbb{R}$.

First, for the linear inhomogeneous differential equations

$$x'(t) = A_{\alpha-1}x(t) + h(t), \quad t \in \mathbb{R}. \tag{3.5}$$

Lemma 3.1. *Let $h \in WPAA(\mathbb{R}, X_{\alpha-1}, \rho)$, then* (3.5) *has a unique mild solution $x \in WPAA(\mathbb{R}, X_\beta, \rho)$ given by*

$$x(t) = \int_{-\infty}^t T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h(\tau)d\tau - \int_t^\infty T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h(\tau)d\tau, \quad t \in \mathbb{R}.$$

Proof. Similarly as the proof in [2], it is clear that $x \in BC(\mathbb{R}, X_\beta)$ and x is a mild solution of (3.5). Since $h \in WPAA(\mathbb{R}, X_{\alpha-1}, \rho)$, let $h = h_1 + h_2$, where $h_1 \in AA(\mathbb{R}, X_{\alpha-1})$, $h_2 \in PAA_0(\mathbb{R}, X_{\alpha-1}, \rho)$. Then $x(t) := x_1(t) + x_2(t)$, where

$$x_1(t) = \int_{-\infty}^t T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h_1(\tau)d\tau - \int_t^\infty T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h_1(\tau)d\tau, \quad t \in \mathbb{R},$$

$$x_2(t) = \int_{-\infty}^t T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h_2(\tau)d\tau - \int_t^\infty T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h_2(\tau)d\tau, \quad t \in \mathbb{R}.$$

Let $(s'_n)_{n\in\mathbb{N}}$ be any sequence of real numbers, then $h_1 \in AA(\mathbb{R}, X_{\alpha-1})$ implies that there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s'_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} h_1(t+s_n) = k(t), \quad \lim_{n\to\infty} k(t-s_n) = h_1(t), \text{ in } X_{\alpha-1} \text{ for } t \in \mathbb{R}.$$

Define

$$H(t) = \int_{-\infty}^t T_{\alpha-1}(t-\tau)P_{s,\alpha-1}k(\tau)d\tau - \int_t^\infty T_{\alpha-1}(t-\tau)P_{u,\alpha-1}k(\tau)d\tau.$$

Let $0 < \tilde{\varepsilon} + \beta < \alpha$ and $0 < \alpha - \tilde{\varepsilon} < 1$, by Lemma 2.1,

$$\begin{aligned}
0 &\leq \|x_1(t+s_n) - H(t)\|_\beta \\
&= \| \int_{-\infty}^{t+s_n} T_{\alpha-1}(t+s_n-\tau)P_{s,\alpha-1}h_1(\tau)d\tau - \int_{t+s_n}^\infty T_{\alpha-1}(t+s_n-\tau)P_{u,\alpha-1}h_1(\tau)d\tau \\
&\quad - \int_{-\infty}^t T_{\alpha-1}(t-\tau)P_{s,\alpha-1}k(\tau)d\tau + \int_t^\infty T_{\alpha-1}(t-\tau)P_{u,\alpha-1}k(\tau)d\tau\|_\beta \\
&\leq \| \int_{-\infty}^{t+s_n} T_{\alpha-1}(t+s_n-\tau)P_{s,\alpha-1}h_1(\tau)d\tau - \int_{-\infty}^t T_{\alpha-1}(t-\tau)P_{s,\alpha-1}k(\tau)d\tau\|_\beta \\
&\quad + \| \int_{t+s_n}^\infty T_{\alpha-1}(t+s_n-\tau)P_{u,\alpha-1}h_1(\tau)d\tau - \int_t^\infty T_{\alpha-1}(t-\tau)P_{u,\alpha-1}k(\tau)d\tau\|_\beta \\
&\leq \| \int_{-\infty}^t T_{\alpha-1}(t-\tau)P_{s,\alpha-1}[h_1(\tau+s_n) - k(\tau)]d\tau\|_\beta
\end{aligned}$$

$$
\begin{aligned}
&+ \| \int_t^{\infty} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}[h_1(\tau+s_n)-k(\tau)]d\tau\|_\beta \\
&\le m\int_{-\infty}^{t} e^{-\gamma(t-\tau)}(t-\tau)^{-(\beta-\alpha+\tilde{\varepsilon}+1)}\|h_1(\tau+s_n)-k(\tau)\|_{\alpha-1}d\tau \\
&+ c\int_t^{\infty} e^{-\delta(\tau-t)}\|h_1(\tau+s_n)-k(\tau)\|_{\alpha-1}d\tau \\
&\le (m\gamma^{\beta-\alpha+\tilde{\varepsilon}}\Gamma(\alpha-\beta-\tilde{\varepsilon})+\delta^{-1}c)\|h_1(\tau+s_n)-k(\tau)\|_{\alpha-1},
\end{aligned}
$$

where $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt$ is the gamma function. Therefore, by the Lebesgue dominated convergence theorem, $\lim\limits_{n\to\infty}\|x_1(t+s_n)-H(t)\|_\beta=0$ for each $t\in\mathbb{R}$. Similarly, $\lim\limits_{n\to\infty}\|H(t-s_n)-x_1(t)\|_\beta=0$ for each $t\in\mathbb{R}$. So $x_1\in AA(\mathbb{R},X_\beta)$.

To complete the proof, we show that $x_2\in PAA_0(\mathbb{R},X_\beta,\rho)$. In fact, for $T>0$, one has

$$
\begin{aligned}
0 &\le \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|x_2(t)\|_\beta dt \\
&\le \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\left\|\int_{-\infty}^{t}T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h_2(\tau)d\tau\right\|_\beta dt \\
&+ \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\left\|\int_{t}^{\infty}T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h_2(\tau)d\tau\right\|_\beta dt \\
&\le \frac{m}{\mu(T,\rho)}\int_{-T}^{T}\int_{-\infty}^{t}e^{-\gamma(t-\tau)}(t-\tau)^{-(\beta-\alpha+\tilde{\varepsilon}+1)}\rho(t)\|h_2(\tau)\|_{\alpha-1}d\tau dt \\
&+ \frac{c}{\mu(T,\rho)}\int_{-T}^{T}\int_{t}^{\infty}e^{-\delta(\tau-t)}\rho(t)\|h_2(\tau)\|_{\alpha-1}d\tau dt \\
&= \frac{m}{\mu(T,\rho)}\int_{-T}^{T}\int_{0}^{\infty}e^{-\gamma\sigma}\sigma^{-(\beta-\alpha+\tilde{\varepsilon}+1)}\rho(t)\|h_2(t-\sigma)\|_{\alpha-1}d\sigma dt \\
&+ \frac{c}{\mu(T,\rho)}\int_{-T}^{T}\int_{0}^{\infty}e^{-\delta\sigma}\rho(t)\|h_2(t+\sigma)\|_{\alpha-1}d\sigma dt \\
&= m\int_0^\infty e^{-\gamma\sigma}\sigma^{-(\beta-\alpha+\tilde{\varepsilon}+1)}\left(\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|h_2(t-\sigma)\|_{\alpha-1}dt\right)d\sigma \\
&+ c\int_0^\infty e^{-\delta\sigma}\left(\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|h_2(t+\sigma)\|_{\alpha-1}dt\right)d\sigma,
\end{aligned}
$$

Since $\rho\in U_T$, from Lemma 2.3, it follows that $h_2(\cdot-\sigma),h_2(\cdot+\sigma)\in PAA_0(\mathbb{R},X_{\alpha-1},\rho)$ for $s\in\mathbb{R}$, then

$$
\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|h_2(t-\sigma)\|_{\alpha-1}dt=0,\quad \lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|h_2(t+\sigma)\|_{\alpha-1}dt=0,
$$

so by Lebesgue dominated convergence theorem,

$$
\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|x_2(t)\|_\beta dt=0,
$$

then $x_2\in PAA_0(\mathbb{R},X_\beta,\rho)$, hence $x\in WPAA(\mathbb{R},X_\beta,\rho)$. $\square$

For (3.3), by the fixed point theorem, one obtains the following conclusion.

Lemma 3.2. *Let $0 \leq \beta < \alpha$ and $\widetilde{\varepsilon} > 0$ such that $0 < \alpha - \widetilde{\varepsilon} < 1$ and $0 < \beta + \widetilde{\varepsilon} < \alpha$. Assume that $h \in WPAA(\mathbb{R} \times X_\beta, X_{\alpha-1}, \rho), \rho \in U_T$, and satisfies*

$$\|h(t,x) - h(t,y)\|_{\alpha-1} \leq k\|x-y\|_\beta, \qquad x, y \in X, t \in \mathbb{R}.$$

If $k[m\gamma^{\beta-\alpha+\widetilde{\varepsilon}}\Gamma(\alpha-\beta-\widetilde{\varepsilon})+\delta^{-1}c] < 1$, then (3.3) has a unique mild solution $x \in WPAA(\mathbb{R}, X_\beta, \rho)$, which satisfies

$$x(t) = \int_{-\infty}^{t} T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h(\tau, x(\tau))d\tau - \int_{t}^{\infty} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h(\tau, x(\tau))d\tau, \qquad t \in \mathbb{R}.$$

Proof. Define the operator $\Gamma : WPAA(\mathbb{R}, X_\beta, \rho) \to WPAA(\mathbb{R}, X_\beta, \rho)$ by

$$(\Gamma x)(t) = \int_{-\infty}^{t} T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h(\tau, x(\tau))d\tau - \int_{t}^{\infty} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h(\tau, x(\tau))d\tau, \qquad t \in \mathbb{R}.$$

By Lemma 3.1, Γ is well defined.

For $x, y \in WPAA(\mathbb{R}, X_\beta, \rho)$,

$$\begin{aligned}
\|(\Gamma x)(t) - (\Gamma y)(t)\|_\beta &\leq m \int_{-\infty}^{t} e^{-\gamma(t-\tau)}(t-\tau)^{-(\beta-\alpha+\widetilde{\varepsilon}+1)}\|h(\tau, x(\tau)) - h(\tau, y(\tau))\|_{\alpha-1}d\tau \\
&\quad + c\int_{t}^{\infty} e^{-\delta(\tau-t)}\|h(\tau, x(\tau)) - h(\tau, y(\tau))\|_{\alpha-1}d\tau \\
&\leq k[m\gamma^{\beta-\alpha+\widetilde{\varepsilon}}\Gamma(\alpha-\beta-\widetilde{\varepsilon}) + \delta^{-1}c]\|x-y\|_\beta.
\end{aligned}$$

By the Banach contraction mapping principle, Γ has a unique fixed point in $WPAA(\mathbb{R}, X_\beta, \rho)$, which is the unique $WPAA$ mild solution to (3.3). The proof is complete. □

Next, we obtain the main result of this section.

Theorem 3.1. *Let $0 \leq \beta < \alpha$ and $\widetilde{\varepsilon} > 0$ such that $0 < \alpha - \widetilde{\varepsilon} < 1$ and $0 < \beta + \widetilde{\varepsilon} < \alpha$. Assume that (H_1)-(H_5) are satisfied, the functions $f \in WPAA(\mathbb{R} \times X_\beta, X, \rho)$, $g \in WPAA(\mathbb{R} \times X_\beta, \partial X, \rho)$ are globally Lipschitzian with small constants. Then (3.1) has a unique mild $x \in WPAA(\mathbb{R}, X_\beta, \rho)$.*

Proof. It is clear that $A_{\alpha-1}L_0$ is a bounded operator from ∂X to $X_{\alpha-1}$. Hence the function $h(t,x) := f(t,x) - A_{\alpha-1}L_0 g(t,x) \in WPAA(\mathbb{R}\times X_\beta, X_{\alpha-1}, \rho)$ and $h(t,x)$ is globally Lipschitzian with a small constant. Hence by (3.2) and Lemma 3.2, there exists a unique mild solution $x \in WPAA(\mathbb{R}, X_\beta, \rho)$ of (3.1).

□

3.2 Weighted Stepanov-like Pseudo Almost Automorphic Perturbation

In this subsection, we deal with the case that the nonlinear perturbation in (3.1) is weighted S^p-pseudo almost automorphic, i.e., the following condition is satisfied:

(H_5') $f \in S^pWPAA(\mathbb{R} \times X_\beta, X, \rho), g \in S^pWPAA(\mathbb{R} \times X_\beta, \partial X, \rho), \rho \in U_T$ for $0 \leq \beta < \alpha$.

For (3.5), one obtains the following results.

Lemma 3.3. *Let $h \in S^pWPAA(\mathbb{R}, X_{\alpha-1}, \rho)$, then* (3.5) *has a unique mild solution $x \in WPAA(\mathbb{R}, X_\beta, \rho)$ given by*

$$x(t) = \int_{-\infty}^{t} T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h(\tau)d\tau - \int_{t}^{\infty} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h(\tau)d\tau, \qquad t \in \mathbb{R}.$$

Proof. Let $h(t) = h_1(t)+h_2(t)$, where $h_1^b \in AA(\mathbb{R}, L^p([0,1], X_{\alpha-1}))$ and $h_2^b \in PAA_0(\mathbb{R}, L^p([0,1], X_{\alpha-1}), \rho)$. Consider the integrals

$$\begin{aligned} v_n(t) &= \int_{t-n}^{t-n+1} T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h(\tau)d\tau - \int_{t+n-1}^{t+n} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h(\tau)d\tau \\ &:= X_n(t) + Y_n(t), \qquad n \in \mathbb{N}, \end{aligned}$$

where

$$X_n(t) = \int_{t-n}^{t-n+1} T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h_1(\tau)d\tau - \int_{t+n-1}^{t+n} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h_1(\tau)d\tau,$$

$$Y_n(t) = \int_{t-n}^{t-n+1} T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h_2(\tau)d\tau - \int_{t+n-1}^{t+n} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h_2(\tau)d\tau.$$

Using (2.2), (2.3) and the Hölder inequality, it follows that

$$\begin{aligned} \|X_n\|_\beta &\le \int_{t-n}^{t-n+1} \|T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h_1(\tau)\|_\beta d\tau + \int_{t+n-1}^{t+n} \|T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h_1(\tau)\|_\beta d\tau \\ &\le m \int_{t-n}^{t-n+1} e^{-\gamma(t-\tau)}(t-\tau)^{-(\beta-\alpha+\tilde{\varepsilon}+1)}\|h_1(\tau)\|_{\alpha-1}d\tau + c\int_{t+n-1}^{t+n} e^{-\delta(\tau-t)}\|h_1(\tau)\|_{\alpha-1}d\tau \\ &\le m \left(\int_{t-n}^{t-n+1} e^{-q\gamma(t-\tau)}(t-\tau)^{-q(\beta-\alpha+\tilde{\varepsilon}+1)}d\tau \right)^{\frac{1}{q}} \left(\int_{t-n}^{t-n+1} \|h_1(\tau)\|_{\alpha-1}^p d\tau \right)^{\frac{1}{p}} \\ &\quad + c\left(\int_{t+n-1}^{t+n} e^{-q\delta(\tau-t)}d\tau \right)^{\frac{1}{q}} \left(\int_{t+n-1}^{t+n} \|h_1(\tau)\|_{\alpha-1}^p d\tau \right)^{\frac{1}{p}} \\ &\le m\|h_1\|_{\alpha-1,S^p} \left(\int_{n-1}^{n} e^{-q\gamma\sigma}\sigma^{-q(\beta-\alpha+\tilde{\varepsilon}+1)}d\sigma \right)^{\frac{1}{q}} + c\|h_1\|_{\alpha-1,S^p}\left(\int_{n-1}^{n} e^{-q\delta\sigma}d\sigma \right)^{\frac{1}{q}}, \qquad (3.6) \end{aligned}$$

where $\|h_1\|_{\alpha-1,S^p} = \sup\limits_{t\in\mathbb{R}} \left(\int_t^{t+1} \|h_1(\tau)\|_{\alpha-1}^p \mathrm{d}\tau \right)^{\frac{1}{p}}$.

Let $\eta := \sum_{n=1}^{\infty}\left(\int_{n-1}^{n} e^{-q\gamma\sigma}\sigma^{-q(\beta-\alpha+\tilde{\varepsilon}+1)}d\sigma\right)^{\frac{1}{q}}$, $\tilde{\eta} := \sum_{n=1}^{\infty}\left(\int_{n-1}^{n} e^{-q\delta\sigma}d\sigma\right)^{\frac{1}{q}}$, then

$$
\begin{aligned}
\eta &= \left(\int_{0}^{1} e^{-q\gamma\sigma}\sigma^{-q(\beta-\alpha+\tilde{\varepsilon}+1)}d\sigma\right)^{\frac{1}{q}} + \sum_{n=2}^{\infty}\left(\int_{n-1}^{n} e^{-q\gamma\sigma}\sigma^{-q(\beta-\alpha+\tilde{\varepsilon}+1)}d\sigma\right)^{\frac{1}{q}} \\
&\leq \varpi + \sum_{n=2}^{\infty}\left(\int_{n-1}^{n} e^{-q\gamma\sigma}d\sigma\right)^{\frac{1}{q}} \leq \varpi + \frac{1}{\sqrt[q]{q\gamma}}\sum_{n=2}^{\infty}\left(e^{-q\gamma(n-1)} - e^{-q\gamma n}\right)^{\frac{1}{q}} \\
&\leq \varpi + \sqrt[q]{(e^{q\gamma}-1)/q\gamma}\sum_{n=2}^{\infty} e^{-\gamma n} \leq \varpi + \sqrt[q]{(e^{q\gamma}+1)/q\gamma}\sum_{n=1}^{\infty} e^{-\gamma n},
\end{aligned}
\tag{3.7}
$$

where $\varpi = \left(\int_{0}^{1} e^{-q\gamma\sigma}\sigma^{-q(\beta-\alpha+\tilde{\varepsilon}+1)}d\sigma\right)^{\frac{1}{q}}$, and

$$
\tilde{\eta} = \frac{1}{\sqrt[q]{q\delta}}\sum_{n=1}^{\infty}\left(e^{-q\delta(n-1)} - e^{-q\delta n}\right)^{\frac{1}{q}} = \sqrt[q]{(e^{q\delta}-1)/q\delta}\sum_{n=1}^{\infty} e^{-\delta n} \leq \sqrt[q]{(e^{q\delta}+1)/q\delta}\sum_{n=1}^{\infty} e^{-\delta n}. \tag{3.8}
$$

Since the series $\sqrt[q]{(e^{q\gamma}+1)/q\gamma}\sum_{n=1}^{\infty} e^{-\gamma n}$, $\sqrt[q]{(e^{q\delta}+1)/q\delta}\sum_{n=1}^{\infty} e^{-\delta n}$ are convergent, by the Weierstrass test, $\sum_{n=1}^{\infty} X_n(t)$ is uniformly convergent on $\mathbb{R}$. Let $X(t) = \sum_{n=1}^{\infty} X_n(t), t \in \mathbb{R}$, then

$$
X(t) = \int_{-\infty}^{t} T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h_1(\tau)d\tau - \int_{t}^{\infty} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h_1(\tau)d\tau.
$$

Fix $n \in \mathbb{N}$ and $t \in \mathbb{R}$, one has

$$
\begin{aligned}
0 &\leq \|X_n(t+\varepsilon) - X_n(t)\|_{\beta} \\
&\leq \left\|\int_{t-n}^{t-n+1} T_{\alpha-1}(t-\tau)P_{s,\alpha-1}[h_1(\tau+\varepsilon) - h_1(\tau)]d\tau\right\|_{\beta} \\
&+ \left\|\int_{t+n-1}^{t+n} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}[h_1(\tau+\varepsilon) - h_1(\tau)]d\tau\right\|_{\beta} \\
&\leq \int_{t-n}^{t-n+1} \|T_{\alpha-1}(t-\tau)P_{s,\alpha-1}[h_1(\tau+\varepsilon) - h_1(\tau)]\|_{\beta}\, d\tau \\
&+ \int_{t+n-1}^{t+n} \|T_{\alpha-1}(t-\tau)P_{u,\alpha-1}[h_1(\tau+\varepsilon) - h_1(\tau)]\|_{\beta}\, d\tau \\
&\leq \int_{t-n}^{t-n+1} me^{-\gamma(t-\tau)}(t-\tau)^{-(\beta-\alpha+\tilde{\varepsilon}+1)}\|h_1(\tau+\varepsilon) - h_1(\tau)\|_{\alpha-1}d\tau
\end{aligned}
$$

$$+c \int_{t+n-1}^{t+n} e^{-\delta(\tau-t)} \|h_1(\tau+\varepsilon)-h_1(\tau)\|_{\alpha-1} d\tau$$

$$\le m \left(\int_{t-n}^{t-n+1} e^{-q\gamma(t-\tau)} (t-\tau)^{-q(\beta-\alpha+\tilde{\varepsilon}+1)} d\tau \right)^{\frac{1}{q}} \left(\int_{t-n}^{t-n+1} \|h_1(\tau+\varepsilon)-h_1(\tau)\|_{\alpha-1}^p d\tau \right)^{\frac{1}{p}}$$

$$+c \left(\int_{t+n-1}^{t+n} e^{-q\delta(\tau-t)} d\tau \right)^{\frac{1}{q}} \left(\int_{t+n-1}^{t+n} \|h_1(\tau+\varepsilon)-h_1(\tau)\|_{\alpha-1}^p d\tau \right)^{\frac{1}{p}},$$

In view of $h_1 \in L^p_{loc}(\mathbb{R}, X_{\alpha-1})$, one has

$$\lim_{\varepsilon\to\infty} \|X_n(t+\varepsilon)-X_n(t)\|_\beta = 0,$$

this means that $X_n \in C(\mathbb{R}, X_\beta)$. Moreover, for any $t \in \mathbb{R}$, from (3.6), (3.7), (3.8), we have

$$\|X_n(t)\|_\beta \le \sum_{n=1}^{\infty} \|X_n(t)\|_\beta \le m\eta \|h_1\|_{\alpha-1,S^p} + c\tilde{\eta}\|h_1\|_{\alpha-1,S^p} < \infty.$$

Next, we prove that $X_n \in AA(\mathbb{R}, X_\beta)$. Since $h_1^b \in AA(\mathbb{R}, L^p([0,1], X_{\alpha-1}))$, then there exist a subsequence $(s_{m_k})_{k\in\mathbb{N}}$ and a function $v_1 \in L^p_{loc}(\mathbb{R}, X)$ such that, for any $t \in \mathbb{R}$,

$$\left(\int_t^{t+1} \|h_1(s+s_{m_k}) - v_1(s)\|_{\alpha-1}^p \, ds \right)^{\frac{1}{p}} \to 0 \quad \text{as} \quad k \to \infty,$$

for any $t \in \mathbb{R}$. Note that

$$\begin{aligned} X_n(t) &= \int_{t-n}^{t-n+1} T_{\alpha-1}(t-\tau) P_{s,\alpha-1} h_1(\tau) d\tau - \int_{t+n-1}^{t+n} T_{\alpha-1}(t-\tau) P_{u,\alpha-1} h_1(\tau) d\tau \\ &= \int_{n-1}^{n} T_{\alpha-1}(\sigma) P_{s,\alpha-1} h_1(t-\sigma) d\sigma - \int_{-n}^{-n+1} T_{\alpha-1}(\sigma) P_{u,\alpha-1} h_1(t-\sigma) d\sigma, \end{aligned}$$

and define

$$w_n(t) := \int_{n-1}^{n} T_{\alpha-1}(\sigma) P_{s,\alpha-1} v_1(t-\sigma) d\sigma - \int_{-n}^{-n+1} T_{\alpha-1}(\sigma) P_{u,\alpha-1} v_1(t-\sigma) d\sigma,$$

then, by Hölder inequality and (2.2), (2.3), we have

$$0 \le \|X_n(t+s_{m_k}) - w_n(t)\|_\beta$$

$$\le \int_{n-1}^{n} \|T_{\alpha-1}(\sigma) P_{s,\alpha-1} [h_1(t-\sigma+s_{m_k}) - v_1(t-\sigma)]\|_\beta d\sigma$$

$$+\int_{-n}^{-n+1} \|T_{\alpha-1}(\sigma)P_{u,\alpha-1}[h_1(t-\sigma+s_{m_k})-v_1(t-\sigma)]\|_\beta d\sigma$$

$$\leq \int_{n-1}^{n} me^{-\gamma\sigma}\sigma^{-(\beta-\alpha+\tilde{\varepsilon}+1)}\|h_1(t-\sigma+s_{m_k})-v_1(t-\sigma)\|_{\alpha-1}d\sigma$$

$$+\int_{-n}^{-n+1} ce^{\delta\sigma}\|h_1(t-\sigma+s_{m_k})-v_1(t-\sigma)\|_{\alpha-1}d\sigma$$

$$\leq m\left(\int_{n-1}^{n} e^{-q\gamma\sigma}\sigma^{-q(\beta-\alpha+\tilde{\varepsilon}+1)}d\sigma\right)^{\frac{1}{q}}\left(\int_{n-1}^{n} \|h_1(t-\sigma+s_{m_k})-v_1(t-\sigma)\|_{\alpha-1}^{p}d\sigma\right)^{\frac{1}{p}}$$

$$+c\left(\int_{-n}^{-n+1} e^{q\delta\sigma}d\sigma\right)^{\frac{1}{q}}\left(\int_{-n}^{-n+1} \|h_1(t-\sigma+s_{m_k})-v_1(t-\sigma)\|_{\alpha-1}^{p}d\sigma\right)^{\frac{1}{p}}$$

$$\leq m\vartheta\left(\int_{n-1}^{n} \|h_1(t-\sigma+s_{m_k})-v_1(t-\sigma)\|_{\alpha-1}^{p}d\sigma\right)^{\frac{1}{p}}$$

$$+c\widetilde{\vartheta}\left(\int_{-n}^{-n+1} \|h_1(t-\sigma+s_{m_k})-v_1(t-\sigma)\|_{\alpha-1}^{p}d\sigma\right)^{\frac{1}{p}},$$

where $\vartheta := \left(\int_{n-1}^{n} e^{-q\gamma\sigma}\sigma^{-q(\beta-\alpha+\tilde{\varepsilon}+1)}d\sigma\right)^{\frac{1}{q}} < \infty$, $\widetilde{\vartheta} := \left(\int_{-n}^{-n+1} e^{q\delta\sigma}d\sigma\right)^{\frac{1}{q}} < \infty$, so

$$\lim_{k\to\infty} \|X_n(t+s_{m_k})-w_n(t)\|_\beta = 0.$$

Similarly, one has $\lim_{k\to\infty} \|w_n(t-s_{m_k})-X_n(t)\|_\beta = 0$, therefore $X_n \in AA(\mathbb{R}, X_\beta)$ for $n \in \mathbb{N}$. By Lemma 2.2, we have $X(t) = \sum_{n=1}^{\infty} X_n(t) \in AA(\mathbb{R}, X_\beta)$.

By carrying out similar arguments as above, we know that $Y_n \in BC(\mathbb{R}, X_\beta)$ and the series $\sum_{n=1}^{\infty} Y_n(t)$ is uniformly convergent on $\mathbb{R}$. Let $Y(t) = \sum_{n=1}^{\infty} Y_n(t)$, then

$$Y(t) = \int_{-\infty}^{t} T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h_2(\tau)d\tau - \int_{t}^{\infty} T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h_2(\tau)d\tau.$$

It is obvious that $Y(t) \in BC(\mathbb{R}, X_\beta)$. So, we only need to show that

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)\|Y(t)\|_\beta \mathrm{d}t = 0.$$

In fact, one has

$$
\begin{aligned}
\|Y_n(t)\|_\beta &\le \int_{t-n}^{t-n+1} \|T_{\alpha-1}(t-\tau)P_{s,\alpha-1}h_2(\tau)\|_\beta d\tau + \int_{t+n-1}^{t+n} \|T_{\alpha-1}(t-\tau)P_{u,\alpha-1}h_2(\tau)\|_\beta d\tau \\
&\le m \int_{t-n}^{t-n+1} e^{-\gamma(t-\tau)}(t-\tau)^{-(\beta-\alpha+\widetilde{\varepsilon}+1)}\|h_2(\tau)\|_{\alpha-1}d\tau + c\int_{t+n-1}^{t+n} e^{-\delta(\tau-t)}\|h_2(\tau)\|_{\alpha-1}d\tau \\
&\le m\left(\int_{t-n}^{t-n+1} e^{-q\gamma(t-\tau)}(t-\tau)^{-q(\beta-\alpha+\widetilde{\varepsilon}+1)}d\tau\right)^{\frac{1}{q}}\left(\int_{t-n}^{t-n+1}\|h_2(\tau)\|^p_{\alpha-1}d\tau\right)^{\frac{1}{p}} \\
&\quad + c\left(\int_{t+n-1}^{t+n} e^{-q\delta(\tau-t)}d\tau\right)^{\frac{1}{q}}\left(\int_{t+n-1}^{t+n}\|h_2(\tau)\|^p_{\alpha-1}d\tau\right)^{\frac{1}{p}} \\
&\le m\vartheta\left(\int_{t-n}^{t-n+1}\|h_2(\tau)\|^p_{\alpha-1}d\tau\right)^{\frac{1}{p}} + c\widetilde{\vartheta}\left(\int_{t+n-1}^{t+n}\|h_2(\tau)\|^p_{\alpha-1}d\tau\right)^{\frac{1}{p}},
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|Y_n(t)\|_\beta dt &\le \frac{m\vartheta}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\left(\int_{t-n}^{t-n+1}\|h_2(\tau)\|^p_{\alpha-1}d\tau\right)^{\frac{1}{p}}dt \\
&\quad + \frac{c\widetilde{\vartheta}}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\left(\int_{t+n-1}^{t+n}\|h_2(\tau)\|^p_{\alpha-1}d\tau\right)^{\frac{1}{p}}dt,
\end{aligned}
$$

and hence $Y_n(t) \in PAA_0(\mathbb{R}, X_\beta, \rho)$ since $h_2^b \in PAA_0(\mathbb{R}, L^p([0,1], X_{\alpha-1}), \rho)$. From $Y_n(t) \in PAA_0(\mathbb{R}, X_\beta, \rho)$ and

$$
\begin{aligned}
\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|Y(t)\|_\beta dt &\le \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|Y(t) - \sum_{n=1}^{N}Y_n(t)\|_\beta dt \\
&\quad + \sum_{n=1}^{N}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|Y_n(t)\|_\beta dt,
\end{aligned}
$$

it follows that $Y(t) \in PAA_0(\mathbb{R}, X_\beta, \rho)$. Therefore, $x \in WPAA(\mathbb{R}, X_\beta, \rho)$. □

By Lemma 3.3 and similarly as the proof Theorem 3.1, one has

Theorem 3.2. *Let $0 \le \beta < \alpha$ and $\widetilde{\varepsilon} > 0$ such that $0 < \alpha - \widetilde{\varepsilon} < 1$ and $0 < \beta + \widetilde{\varepsilon} < \alpha$. Assume that (H_1)-(H_4) and (H_5') are satisfied, the functions $f \in S^pWPAA(\mathbb{R} \times X_\beta, X, \rho)$, $g \in S^pWPAA(\mathbb{R} \times X_\beta, \partial X, \rho)$ are globally Lipschitzian with small constants. Then (3.1) has a unique mild $x \in WPAA(\mathbb{R}, X_\beta, \rho)$.*

4 Examples

Consider the following partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + au(t,x), & t \in \mathbb{R}, x \in \Omega \\ \dfrac{\partial}{\partial n}u(t,x) = \Psi(t, m(x)u(t,x)), & t \in \mathbb{R}, x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $a \in \mathbb{R}^+$ and m is a C^1-function, Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

Let $X = L^2(\Omega)$, $X_m = H^2(\Omega)$ and the boundary space $\partial X = H^{\frac{1}{2}}(\partial\Omega)$. Consider the operator $A_m : X_m \to X$ $A_m\varphi = \Delta\varphi + a\varphi$ and $L : X_m \to \partial X$, $L\varphi = \frac{\partial\varphi}{\partial n}$. By [2], the operator $A = A_m|_{kerL}$ generates an analytic semigroup, and for $\alpha < \frac{3}{4}$, $X_m \subset X_\alpha$. The eigenvalues of the operator A is a decreasing sequence (λ_n) with $\lambda_0 = 1$ and $\lambda_1 < 0$. If one takes $a = \frac{1}{2}\lambda_1$, then $\sigma(A) \cap i\mathbb{R} = \emptyset$, so the analytic semigroup generated by A is hyperbolic.

Let $\phi(t,\varphi)(x) = \Psi(t, m(x)u(t,x)) = \frac{kb(t)}{1+|m(x)\varphi(x)|}$, $t \in \mathbb{R}, x \in \partial\Omega$ and $b(t) \in WPAA(\mathbb{R}, \Omega)$. One can see that ϕ is continuous on $\mathbb{R} \times H^{2\beta'}(\Omega)$ for some $\frac{1}{2} < \beta < \beta' < \alpha < \frac{3}{4}$ which is embedded in $\mathbb{R} \times X_\beta$. It is not difficult to see that $\phi(t,\varphi)(\cdot) \in H^{\frac{1}{2}}(\partial\Omega, \partial\Omega)$ for all $\varphi \in H^{2\beta'}(\Omega) \hookrightarrow H^1(\Omega)$. Moreover, ϕ is weighted pseudo almost automorphic in $t \in \mathbb{R}$ for each $\varphi \in X_\beta$, and globally Lipschitzian. Then for a small constant k, (4.1) exists a unique weighted pseudo almost automorphic mild solution $u \in X_\beta$.

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